

ON THE R -BOUNDEDNESS OF STOCHASTIC CONVOLUTION OPERATORS

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ABSTRACT. The R -boundedness of certain families of vector-valued stochastic convolution operators with scalar-valued square integrable kernels is the key ingredient in the recent proof of stochastic maximal L^p -regularity, $2 < p < \infty$, for certain classes of sectorial operators acting on spaces $X = L^q(\mu)$, $2 \leq q < \infty$. This paper presents a systematic study of R -boundedness of such families. Our main result generalises the afore-mentioned R -boundedness result to a larger class of Banach lattices X and relates it to the ℓ^1 -boundedness of an associated class of deterministic convolution operators. We also establish an intimate relationship between the ℓ^1 -boundedness of these operators and the boundedness of the X -valued maximal function. This analysis leads, quite surprisingly, to an example showing that R -boundedness of stochastic convolution operators fails in certain UMD Banach lattices with type 2.

1. INTRODUCTION

Maximal L^p -regularity is a tool of central importance in the theory of parabolic PDEs, as it enables one to reduce the study of various classes of ‘complicated’ non-linear PDEs to a fixed point problem, e.g. by linearisation (see [2, 8, 19] and the references therein). The extension of this circle of ideas to parabolic stochastic PDE required new ideas and was achieved only recently in [28], where it was shown that if a sectorial operator A admits a bounded H^∞ -calculus of angle less than $\pi/2$ on a space $L^q(D, \mu)$, with $q \in [2, \infty)$ and (D, μ) a σ -finite measure space, then for all Hilbert spaces H and adapted processes $G \in L^p(\mathbb{R}_+ \times \Omega; L^q(D, \mu; H))$ the stochastic convolution process

$$U(t) = \int_0^t e^{-(t-s)A} G(s) dW_H(s), \quad t \geq 0,$$

with respect to any cylindrical Brownian motion W_H in H , is well-defined in $L^q(D, \mu)$, takes values in the fractional domain $D(A^{1/2})$ almost surely, and satisfies, for $2 < p < \infty$, the stochastic maximal L^p -regularity estimate

$$(1.1) \quad \mathbb{E} \|A^{1/2} U\|_{L^p(\mathbb{R}_+; L^q(D, \mu))}^p \leq C^p \mathbb{E} \|G\|_{L^p(\mathbb{R}_+; L^q(D, \mu; H))}^p.$$

Applications to semilinear parabolic SPDEs were worked out subsequently in [27]. By now, two proofs of the stochastic maximal L^p -regularity theorem are available:

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the original one of [28] based on H^∞ -calculus techniques combined with the Poisson formula for holomorphic functions on an open sector in the complex plane, and a second one based on operator-valued H^∞ -calculus techniques [29]. Both proofs, however, critically depend upon the R -boundedness of a suitable class of vector-valued stochastic convolution operators with scalar-valued kernels. For stochastic convolution operators taking values in a space $L^q(\mu)$ with $2 \leq q < \infty$, the R -boundedness of this family has been derived in [28] as a consequence of the Fefferman-Stein theorem on the $L^p(L^q(\mu))$ -boundedness of the Hardy-Littlewood maximal function; it is for this reason that the theory, in its present state, is essentially limited to SPDEs with state space $X = L^q(\mu)$.

The aim of this paper is to undertake a systematic analysis of the R -boundedness properties of families of stochastic convolution operators with scalar-valued square integrable kernels k taking values in an arbitrary Banach lattice X . The main result asserts that such a family is R -bounded if and only if the corresponding family of *deterministic* convolution operators corresponding to the squared kernels k^2 is ℓ^1 -bounded. The notion of ℓ^s -boundedness (also called R^s -boundedness), $1 \leq s \leq \infty$, has been introduced in [42] and was systematically studied in [18, 40]. For operators acting on Banach lattices X with finite cotype, R -boundedness is equivalent to ℓ^2 -boundedness. Moreover, in [20] it is shown that this can only be true if X has finite cotype.

Thus the problem of stochastic maximal L^p -regularity is reduced to the problem of ℓ^1 -boundedness of suitable families of deterministic convolution operators with integrable kernels. Our second main result establishes the ℓ^1 -boundedness of such operators under the assumption that X is a Banach lattice with type 2 with the additional property that the dual of its 2-convexification has the so-called Hardy-Littlewood property, meaning essentially that the Fefferman-Stein theorem holds for this space. A sufficient condition for the latter is that the 2-convexification is a UMD Banach function space. In [37, Theorem 3], the same condition was shown to imply the X -valued Littlewood-Paley-Rubio de Francia property.

In Section 8 we show that the Banach lattice $\ell^\infty(\ell^2) = (\ell^1(\ell^2))^*$ fails the Hardy-Littlewood property (see Definition 4.1 below), and for this reason $X = \ell^2(\ell^4)$ (whose 2-concavification equals $\ell^1(\ell^2)$) is a natural candidate of a Banach lattice in which R -boundedness of X -valued stochastic convolution operators might fail. In the final section of this paper we establish our third main result, which turns this suspicion into a theorem. The failure of R -boundedness of stochastic convolutions in $\ell^2(\ell^4)$ is quite remarkable, as this space is a UMD Banach lattice with type 2.

2. PRELIMINARIES

Throughout this paper, all vector spaces are real. In this preliminary section we collect some results that will be needed in the sequel.

2.1. R -boundedness. (See [8, 19]). Let X and Y be real Banach spaces and let $(r_n)_{n \geq 1}$ be a *Rademacher sequence* on a probability space (Ω, \mathbb{P}) , that is, a sequence of independent random variables $r_n : \Omega \rightarrow \{-1, 1\}$ taking the values ± 1 with probability $\frac{1}{2}$. A family \mathcal{T} of bounded linear operators from X to Y is called *R -bounded* if there exists a constant $C \geq 0$ such that for all finite sequences $(T_n)_{n=1}^N$

in \mathcal{T} and $(x_n)_{n=1}^N$ in X we have

$$\mathbb{E} \left\| \sum_{n=1}^N r_n T_n x_n \right\|^2 \leq C^2 \mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^2.$$

The least admissible constant C is called the R -bound of \mathcal{T} , notation $R(\mathcal{T})$.

2.2. Spaces of radonifying operators. (See [25]). Let H be a Hilbert space and X a Banach space. For $h \in H$ and $x \in X$ we denote by $h \otimes x$ the rank one operator from H to X given by $h' \mapsto [h', h]x$. Let $(\gamma_n)_{n \geq 1}$ be a Gaussian sequence defined on some probability space (Ω, \mathbb{P}) . The γ -radonifying norm of a finite rank operator of the form $\sum_{n=1}^N h_n \otimes x_n$, where the vectors h_1, \dots, h_N are orthonormal in H and x_1, \dots, x_N are taken from X , is defined by

$$(2.1) \quad \left\| \sum_{n=1}^N h_n \otimes x_n \right\|_{\gamma(H, X)}^2 := \mathbb{E} \left\| \sum_{n=1}^N \gamma_n x_n \right\|^2.$$

The invariance of standard Gaussian vectors in \mathbb{R}^n under orthogonal transformations easily implies that this is well defined. The completion of the space $H \otimes X$ of all finite rank operators from H into X with respect to the norm $\|\cdot\|_{\gamma(H, X)}$ is denoted by $\gamma(H, X)$. This space is continuously and contractively embedded in $\mathcal{L}(H, X)$. A bounded operator in $\mathcal{L}(H, X)$ is said to be γ -radonifying if it belongs to $\gamma(H, X)$. If H is separable, say with orthonormal basis $(h_n)_{n \geq 1}$, then an operator $T \in \mathcal{L}(H, X)$ is γ -radonifying if and only if the sum $\sum_{n \geq 1} \gamma_n T h_n$ converges in $L^2(\Omega; X)$, and in this case we have

$$\|T\|_{\gamma(H, X)}^2 = \mathbb{E} \left\| \sum_{n \geq 1} \gamma_n T h_n \right\|^2.$$

The space $\gamma(H, X)$ is an *operator ideal* in $\mathcal{L}(H, X)$ in the sense that if $S_1 : \tilde{H} \rightarrow H$ and $S_2 : X \rightarrow \tilde{X}$ are bounded operators, then $T \in \gamma(H, X)$ implies $S_2 T S_1 \in \gamma(\tilde{H}, \tilde{X})$ and

$$(2.2) \quad \|S_2 T S_1\|_{\gamma(\tilde{H}, \tilde{X})} \leq \|S_2\| \|T\|_{\gamma(H, X)} \|S_1\|.$$

Let $p \in [1, \infty)$ be given, let (Ω, \mathbb{P}) be a probability space, and suppose that $W : L^2(\mathbb{R}_+; H) \rightarrow L^2(\Omega)$ is an H -cylindrical Brownian motion (see Section 6 for the precise definition). Then the stochastic integral $h \otimes x \mapsto W h \otimes x$ extends to an isomorphic embedding of $\gamma(L^2(\mathbb{R}_+; H), X)$ onto a closed subspace of $L^p(\Omega; X)$. This fact will be used in the proof of Proposition 5.2; a more detailed account of stochastic integration with respect to cylindrical Brownian motion will be given in Section 6.

Example 2.1. If X is a Hilbert space, then $\gamma(H, X)$ is isometrically isomorphic to the Hilbert space of Hilbert–Schmidt operators from H to X . If (S, μ) is a σ -finite measure space and $p \in [1, \infty)$, then $\gamma(H, L^p(S)) = L^p(S; H)$ with equivalent norms, the isomorphism being given by associating to the function $f \in L^p(S; H)$ the mapping $h \mapsto [f(\cdot), h]_H$ from H to $L^p(S)$.

More generally we have (see [26, Proposition 2.6]) :

Proposition 2.2 (γ -Fubini isomorphism). *For any Banach space X the mapping $h \otimes (f \otimes x) \mapsto f \otimes (h \otimes x)$ extends by linearity to an isomorphism of Banach spaces*

$$\gamma(H, L^p(\mathbb{R}^d; X)) \approx L^p(\mathbb{R}^d; \gamma(H, X)).$$

The next simple proposition extends [23, Proposition 6.2]. For $f \in L^p(S)$ and $y \in Y$ we denote by $f \otimes y$ the function in $L^p(S; Y)$ defined by $(f \otimes y)(s) := f(s)y$. By I_H we denote the identity operator on H .

Proposition 2.3. *Let \mathcal{T} be an R -bounded family of bounded linear operators from X to Y and let H be a nonzero Hilbert space. Then the family $I_H \otimes \mathcal{T} = \{I_H \otimes T : T \in \mathcal{T}\}$ is R -bounded from $\gamma(H, X)$ to $\gamma(H, Y)$ and $R(I_H \otimes \mathcal{T}) = R(\mathcal{T})$.*

Proof. Fix $T_1, \dots, T_N \in \mathcal{T}$ and $R_1, \dots, R_N \in \gamma(H, X)$. Since each R_n is the limit of at most countably many finite rank operators we may assume that H is separable. Let $(h_m)_{m \geq 1}$ be an orthonormal basis for H . Then,

$$\begin{aligned} \mathbb{E}_r \left\| \sum_{n=1}^N r_n (I_H \otimes T_n) R_n \right\|_{\gamma(H, Y)}^2 &= \mathbb{E}_r \mathbb{E}_\gamma \left\| \sum_{n=1}^N \sum_{m \geq 1} r_n \gamma_m T_n R_n h_m \right\|^2 \\ &\leq R(\mathcal{T})^2 \mathbb{E}_r \mathbb{E}_\gamma \left\| \sum_{n=1}^N \sum_{m \geq 1} r_n \gamma_m R_n h_m \right\|^2 \\ &= R(\mathcal{T})^2 \mathbb{E}_r \left\| \sum_{n=1}^N r_n R_n \right\|_{\gamma(H, X)}^2. \end{aligned}$$

This proves the R -boundedness of $I_H \otimes \mathcal{T}$ along with the bound $R(I_H \otimes \mathcal{T}) \leq R(\mathcal{T})$. The converse inequality is trivial. \square

2.3. Type and cotype. (See [9, 21]). A Banach space X has *type* $p \in [1, 2]$ if there exists a constant $C \geq 0$ such that for all $N \geq 1$ and all finite sequences $(x_n)_{n=1}^N$ in X we have

$$\left(\mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^2 \right)^{\frac{1}{2}} \leq C \left(\sum_{n=1}^N \|x_n\|^p \right)^{\frac{1}{p}}.$$

The least admissible constant C is called the *type p constant* of X , notation $T_p(X)$. Similarly, X has *cotype* $q \in [2, \infty]$ if there exists a constant $C \geq 0$ such that for all $N \geq 1$ and all finite sequences $(x_n)_{n=1}^N$ in X we have

$$\left(\sum_{n=1}^N \|x_n\|^q \right)^{\frac{1}{q}} \leq C \left(\mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^2 \right)^{\frac{1}{2}}$$

(with an obvious modification if $q = \infty$). The least admissible constant C is called the *cotype q constant* of X , notation $C_q(X)$.

Every Banach space has type 1 and cotype ∞ , Hilbert spaces have type 2 and cotype 2, and the spaces $L^p(S)$ have type $\min\{p, 2\}$ and cotype $\max\{p, 2\}$ for $p \in [1, \infty)$. All UMD spaces have non-trivial type, i.e., type $p \in (1, 2]$.

Spaces of type 2 are of special importance to us for the following reason.

Proposition 2.4 ([31, 38]). *Let (S, Σ, μ) be a σ -finite measure space, H a non-zero Hilbert space, and X a Banach space.*

(1) *X has type 2 if and only if the mapping*

$$f \otimes (h \otimes x) \mapsto (f \otimes h) \otimes x, \quad f \in L^2(S), \quad h \in H, \quad x \in X,$$

extends to continuous embedding

$$I : L^2(S; \gamma(H, X)) \hookrightarrow \gamma(L^2(S; H), X).$$

In this case we have $\|I\| \leq T_2(X)$.

(2) X has cotype 2 if and only if the mapping

$$(f \otimes h) \otimes x \mapsto f \otimes (h \otimes x), \quad f \in L^2(S), \quad h \in H, \quad x \in X,$$

extends to continuous embedding

$$J : \gamma(L^2(S; H), X) \hookrightarrow L^2(S; \gamma(H, X)).$$

In this case we have $\|J\| \leq C_2(X)$.

2.4. Double Rademacher sums. (See [32]). Let $(r_{mn})_{m,n \geq 1}$ be a doubly indexed Rademacher sequence on a probability space (Ω, \mathbb{P}) and let $(r'_n)_{n \geq 1}$ and $(r''_n)_{n \geq 1}$ be Rademacher sequences on independent probability spaces (Ω', \mathbb{P}') and (Ω'', \mathbb{P}'') respectively.

Definition 2.5 (See [32, 35]). *Let X be a Banach space.*

(1) X has property (α^+) if there is a constant $C^+ \geq 0$ such that for all finite doubly-indexed sequences $(x_{mn})_{m,n=1}^{M,N}$ in X we have

$$\left(\mathbb{E} \left\| \sum_{m=1}^M \sum_{n=1}^N r_{mn} x_{mn} \right\|^2 \right)^{1/2} \leq C^+ \left(\mathbb{E}' \mathbb{E}'' \left\| \sum_{m=1}^M \sum_{n=1}^N r'_m r''_n x_{mn} \right\|^2 \right)^{1/2}.$$

(2) X has property (α^-) if there is a constant $C^- \geq 0$ such that for all finite doubly-indexed sequences $(x_{mn})_{m,n=1}^{M,N}$ in X we have

$$\left(\mathbb{E}' \mathbb{E}'' \left\| \sum_{m=1}^M \sum_{n=1}^N r'_m r''_n x_{mn} \right\|^2 \right)^{1/2} \leq C^- \left(\mathbb{E} \left\| \sum_{m=1}^M \sum_{n=1}^N r_{mn} x_{mn} \right\|^2 \right)^{1/2}.$$

(3) X has property (α) if X has property (α^+) and (α^-) .

Each of the properties (α^+) and (α^-) implies finite cotype, and conversely every Banach lattice with finite cotype has property (α) . The space c_0 fails both (α^+) and (α^-) . For the Schatten class C^p with $p \in [1, \infty)$ one has the following results which follows from the proofs in [36]:

- (i) C^p has property (α^+) if and only if $p \in [2, \infty)$
- (ii) C^p has property (α^-) if and only if $p \in [1, 2]$.

In particular, C^p has property (α) if and only if $p = 2$.

Below we shall need part (1) of the following result.

Proposition 2.6 ([32, Theorem 3.3]). *Let H_1 and H_2 be non-zero Hilbert spaces and denote by $H_1 \overline{\otimes} H_2$ their Hilbert space tensor product. The following assertions hold:*

- (1) X has property (α^+) if and only if the map $h_1 \otimes (h_2 \otimes x) \mapsto (h_1 \otimes h_2) \otimes x$ extends to a bounded operator from $\gamma(H_1, \gamma(H_2, X))$ into $\gamma(H_1 \overline{\otimes} H_2, X)$;
- (2) X has property (α^-) if and only if the map $(h_1 \otimes h_2) \otimes x \mapsto h_1 \otimes (h_2 \otimes x)$ extends to a bounded operator from $\gamma(H_1 \overline{\otimes} H_2, X)$ into $\gamma(H_1, \gamma(H_2, X))$.

The next result establishes a relation between the notions of type and cotype and the properties (α^+) and (α^-) .

Proposition 2.7. *Let X be a Banach space.*

- (1) If X has type 2, then X has property (α^+) .
- (2) If X has cotype 2, then X has property (α^-) .

Proof. (1): Since $(r'_m r''_n)_{m,n \geq 1}$ is an orthonormal system in $L^2(\Omega' \times \Omega'')$, part (1) is a consequence of [11, Theorem 1.3]. A more direct proof, which can be modified to give part (2) as well, runs as follows.

Set $h_{mn} := r'_m r''_n$. Estimating Rademacher sums by Gaussian sums (see [9, Proposition 12.11] or [21, Lemma 4.5]) and using (2.1) and Proposition 2.4(1), we obtain

$$\begin{aligned} \mathbb{E} \left\| \sum_{m=1}^M \sum_{n=1}^N r_{mn} x_{mn} \right\|^2 &\leq \frac{1}{2} \pi \mathbb{E} \left\| \sum_{m=1}^M \sum_{n=1}^N \gamma_{mn} x_{mn} \right\|^2 \\ &= \frac{1}{2} \pi \left\| \sum_{m=1}^M \sum_{n=1}^N h_{mn} \otimes x_{mn} \right\|_{\gamma(L^2(\Omega' \times \Omega''), X)}^2 \\ &\leq \frac{1}{2} \pi (T_2(X))^2 \left\| \sum_{m=1}^M \sum_{n=1}^N h_{mn} \otimes x_{mn} \right\|_{L^2(\Omega' \times \Omega''; X)}^2 \\ &= \frac{1}{2} \pi (T_2(X))^2 \mathbb{E}' \mathbb{E}'' \left\| \sum_{m=1}^M \sum_{n=1}^N r'_m r''_n x_{mn} \right\|^2. \end{aligned}$$

This gives the result.

(2): This is proved in the same way, this time using Proposition 2.4(2) along with the fact that in the presence of finite cotype, Gaussian sums can be estimated by Rademacher sums (see [9, Theorem 12.27] or [21, Proposition 9.14]). \square

Lemma 2.8. *Let X be a Banach function space with finite cotype and let (S, Σ, μ) be a σ -finite measure space. Let $G_n \in L^2(S; X)$, $1 \leq n \leq N$, be functions taking values in a finite-dimensional subspace of X . Then for all $1 \leq p < \infty$ we have*

$$\left(\mathbb{E} \left\| \sum_{n=1}^N r_n G_n \right\|_{\gamma(L^2(S), X)}^p \right)^{1/p} \approx_{p, X} \left\| \left(\sum_{n=1}^N \int_S |G_n|^2 d\mu \right)^{1/2} \right\|.$$

Proof. By the Kahane-Khintchine inequalities it suffices to consider $p = 2$. By [9, Proposition 12.11 and Theorem 12.27],

$$\mathbb{E} \left\| \sum_{n=1}^N r_n G_n \right\|_{\gamma(L^2(S), X)}^2 \approx \mathbb{E} \left\| \sum_{n=1}^N \gamma_n G_n \right\|_{\gamma(L^2(S), X)}^2 = \|G\|_{\gamma(\ell_N^2; \gamma(L^2(S), X))}^2.$$

Moreover, by Proposition 2.6, $\gamma(\ell_N^2, \gamma(L^2(S), X)) \approx \gamma(L^2(S) \otimes \ell_N^2, X)$ isomorphically. Now the result follows from (the proof of) [30, Corollary 2.10]. \square

2.5. The UMD property and martingale type. (See [7, 34, 39]). A Banach space X is called a *UMD space* if for some $p \in (1, \infty)$ (equivalently, for all $p \in (1, \infty)$; see [7]) there is a constant $\beta \geq 0$ such that for all finite X -valued L^p -martingale difference sequences $(d_n)_{n=1}^N$ and sequence of signs $(\varepsilon_n)_{n=1}^N$ one has

$$(2.3) \quad \mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n d_n \right\|^p \leq \beta^p \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p.$$

The least admissible constant in this definition is called the *UMD_p-constant* of X and is denoted by $\beta_{p, X}$. If $(r_n)_{n \geq 1}$ is a Rademacher sequence which is independent

of $(d_n)_{n=1}^N$, then (2.3) and its counterpart applied to the martingales $\varepsilon_n d_n$ easily imply the two-sided randomised inequality

$$(2.4) \quad \frac{1}{\beta_{p,X}^p} \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \leq \mathbb{E} \left\| \sum_{n=1}^N r_n d_n \right\|^p$$

where now $(r_n)_{n \geq 1}$ is a Rademacher sequence independent of $(d_n)_{n=1}^N$.

Examples of UMD spaces include Hilbert spaces and the Lebesgue spaces $L^p(S)$ for $1 < p < \infty$. Noting that every UMD space is reflexive, it follows that $L^\infty(S)$ and $L^1(S)$ are not UMD spaces.

Let $p \in [1, 2]$. A Banach space X has *martingale type p* if there exists a constant $\mu \geq 0$ such that for all finite X -valued martingale difference sequences $(d_n)_{n=1}^N$ we have

$$(2.5) \quad \mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \leq \mu^p \sum_{n=1}^N \mathbb{E} \|d_n\|^p.$$

The least admissible constant in this definition is denoted by $\mu_{p,X}$.

Trivially, martingale type p implies type p . Hilbert spaces have martingale type 2 and every Lebesgue space $L^p(S)$, $1 \leq p < \infty$, has martingale type $p \wedge 2$. In fact we have the following equivalence (see [4]):

Proposition 2.9. *Let $p \in [1, 2]$.*

- (1) *A UMD Banach space X has martingale type p if and only if it has type p .*
- (2) *A Banach lattice X has martingale type 2 if and only if it has type 2.*

Proof. (1): Suppose that X has type p and let $(\tilde{r}_n)_{n \geq 1}$ be a Rademacher sequence on another probability space $(\tilde{\Omega}, \tilde{\mathbb{P}})$. By (2.4) and Fubini's theorem,

$$\mathbb{E} \left\| \sum_{n=1}^N d_n \right\|^p \leq \beta_{p,X}^p \mathbb{E} \tilde{\mathbb{E}} \left\| \sum_{n=1}^N \tilde{r}_n d_n \right\|^p \leq \beta_{p,X}^p \tau_{p,X}^p \mathbb{E} \sum_{n=1}^N \|d_n\|^p.$$

It follows that X has martingale type p .

(2): Suppose that X has type 2. By [22, Theorem 1.f.17], X is 2-convex and q -concave for some $q < \infty$. By [22, Theorem 1.f.1], this implies that X is 2-smooth. Hence by [34], X has martingale type 2. \square

3. ℓ^s -BOUNDEDNESS

For Rademacher sums with values in a Banach lattice X with finite cotype we have the two-sided estimate

$$(3.1) \quad \left(\mathbb{E} \left\| \sum_{n=1}^N r_n x_n \right\|^2 \right)^{1/2} \approx \left(\mathbb{E} \left\| \sum_{n=1}^N \gamma_n x_n \right\|^2 \right)^{1/2} \approx \left\| \left(\sum_{n=1}^N |x_n|^2 \right)^{1/2} \right\|$$

with implied constants depending only on X (see [9, Proposition 12.11, Theorems 12.27 and 16.18]). The expression on the right-hand side acquires its meaning through the so-called Krivine calculus. We refer to [22, Section II.1.d] for a detailed exposition of this calculus. In all applications below, X is a Banach function space and in this special case the expression can be defined in a pointwise sense in the obvious manner. If X and Y are Banach lattices with finite cotype, a family \mathcal{T} of bounded linear operators from X to Y is R -bounded if and only if there is a

constant $C \geq 0$ such that for all finite sequences $(T_n)_{n=1}^N$ in \mathcal{T} and $(x_n)_{n=1}^N$ in X we have

$$\left\| \left(\sum_{n=1}^N |T_n x_n|^2 \right)^{1/2} \right\| \leq C \left\| \left(\sum_{n=1}^N |x_n|^2 \right)^{1/2} \right\|,$$

This motivates the following definition.

Definition 3.1. *Let X and Y be Banach lattices and let $s \in [1, \infty]$. A family of operators $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is called ℓ^s -bounded if there is a constant $C \geq 0$ such that for all finite sequences $(T_n)_{n=1}^N$ in \mathcal{T} and $(x_n)_{n=1}^N$ in X we have*

$$\left\| \left(\sum_{n=1}^N |T_n x_n|^s \right)^{1/s} \right\| \leq C \left\| \left(\sum_{n=1}^N |x_n|^s \right)^{1/s} \right\|,$$

with the obvious modification if $s = \infty$.

The least admissible constant C in Definition 3.1 is called the ℓ^s -bound of \mathcal{T} and is denoted by $R^s(\mathcal{T})$ and usually abbreviated as $R^s(\mathcal{T})$.

The notion of ℓ^s -boundedness was introduced in [42] in the context of the so-called (deterministic) maximal regularity problem; for a systematic treatment we refer the reader to [18, 40].

Example 3.2 ([18, Remark 2.7]). Let (S, μ) be a measure space. For all $s \in [1, \infty]$, the unit ball of $\mathcal{L}(L^s(S))$ is ℓ^s -bounded, with constant $R^s(\mathcal{T}) \leq 1$.

Remark 3.3. Let $p_i \in [1, \infty)$ and let (S_i, μ_i) be a measure space for $i = 1, 2$. Let $T : L^{p_1}(S_1) \rightarrow L^{p_1}(S_2)$ be a bounded operator. It follows from [6, Lemma 1.7] that the singleton $\{T\}$ is ℓ^1 -bounded if and only if T can be written as the difference of two positive operators. In this result one can replace $L^{p_i}(S_i)$ by certain Banach function spaces. This shows that for an operator family \mathcal{T} to be ℓ^1 -bounded imposes a rather special structure on the operators in \mathcal{T} .

Let X be a Banach lattice. We denote by $X(\ell_N^s)$ the Banach space of all sequences $(x_n)_{n=1}^N$ in X endowed with the norm

$$\|(x_n)_{n=1}^N\|_{X(\ell_N^s)} := \left\| \left(\sum_{n=1}^N |x_n|^s \right)^{1/s} \right\|,$$

again with the obvious modification if $s = \infty$. More details on these spaces can be found in [22, p. 47]. Using this terminology, the definition of ℓ^s -boundedness can be rephrased as saying that

$$(3.2) \quad \|(T_n x_n)_{n=1}^N\|_{Y(\ell_N^s)} \leq C \|(x_n)_{n=1}^N\|_{X(\ell_N^s)}$$

for all finite sequences $(T_n)_{n=1}^N$ in \mathcal{T} and $(x_n)_{n=1}^N$ in X .

For $X = \mathbb{R}$ we have $X(\ell_N^s) = \ell_N^s$ canonically for all $s \in [1, \infty]$. For any Banach lattice X the mapping

$$(t \mapsto f_n(t))_{n=1}^N \mapsto (t \mapsto (f_n(t))_{n=1}^N)$$

establishes an isometric isomorphism

$$(3.3) \quad (L^p(S; X))(\ell_N^s) = L^p(S; X(\ell_N^s))$$

for all $p \in [1, \infty]$ and $s \in [1, \infty]$.

The following properties have been stated in [40, Section 3.1]. Recall that every reflexive Banach lattice has order continuous norm (see [24, Section 2.4] for details).

Proposition 3.4. *Let X and Y be Banach lattices and let $s, s_0, s_1 \in [1, \infty]$. Let $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ be a family of bounded operators.*

- (1) *If \mathcal{T} is ℓ^s -bounded, then also its strongly closed absolutely convex hull $\overline{\text{absco}}(\mathcal{T})$ is ℓ^s -bounded and*

$$R^s(\overline{\text{absco}}(\mathcal{T})) = R^s(\mathcal{T}).$$

- (2) *The family \mathcal{T} is ℓ^s -bounded if and only if the adjoint family \mathcal{T}^* is $\ell^{s'}$ -bounded, where $\frac{1}{s} + \frac{1}{s'} = 1$, and in this case we have*

$$R^{s'}(\mathcal{T}^*) = R^s(\mathcal{T}).$$

- (3) *Suppose that $\mathcal{T} \subseteq \mathcal{L}(X, Y)$ is both ℓ^{s_0} -bounded and ℓ^{s_1} -bounded. If at least one of the spaces X or Y has order continuous norm, then \mathcal{T} is ℓ^{s_θ} -bounded for all $\theta \in (0, 1)$, where $\frac{1}{s_\theta} = \frac{1-\theta}{s_0} + \frac{\theta}{s_1}$, and*

$$R^{s_\theta}(\mathcal{T}) \leq (R^{s_0}(\mathcal{T}))^{1-\theta} (R^{s_1}(\mathcal{T}))^\theta.$$

For the proof of (1) one can repeat the analogous argument for R -boundedness (see [19, Theorem 2.13]). Assertion (2) follows from the identification $X(\ell_N^s)^* = X^*(\ell_N^{s'})$ (see [22, p. 47]). Assertion (3) follows by complex interpolation (see [40, pages 57–58]).

4. ℓ^s -BOUNDEDNESS OF CONVOLUTION OPERATORS

If X is a Banach lattice and $J \subseteq \mathbb{R}_+$ is a finite subset, for $f \in L_{\text{loc}}^1(\mathbb{R}^d; X)$ we may define

$$(\widetilde{M}_J f)(\xi) := \sup_{r \in J} \frac{1}{|B_\xi(r)|} \int_{B_\xi(r)} |f(\eta)| d\eta, \quad \xi \in \mathbb{R}^d,$$

where the modulus and supremum are taken in the lattice sense of X .

Definition 4.1. *We say that X has the Hardy-Littlewood property (briefly, X is an HL space) if for all $p \in (1, \infty)$ and $d \geq 1$, all finite subsets $J \subseteq \mathbb{R}_+$, and all $f \in L^p(\mathbb{R}^d; X)$ we have $\widetilde{M}_J f \in L^p(\mathbb{R}^d; X)$ and there is a finite constant $C = C_{p,d,X} \geq 0$, independent of J and f , such that*

$$\|\widetilde{M}_J f\|_{L^p(\mathbb{R}^d; X)} \leq C \|f\|_{L^p(\mathbb{R}^d; X)}, \quad f \in L^p(\mathbb{R}^d; X).$$

In this situation we will say that \widetilde{M} is bounded on $L^p(\mathbb{R}^d; X)$.

In [13] it has been proved that the Hardy-Littlewood property for fixed $p \in (1, \infty)$ and $d \geq 1$ implies the corresponding property for all $p \in (1, \infty)$ and $d \geq 1$, that is, the property is independent of $p \in (1, \infty)$ and $d \geq 1$.

In order to be able to deal with lattice suprema indexed by infinite sets J we need to introduce some terminology. A Banach lattice X is called *monotonically complete* if $\sup_{i \in I} x_i$ exists for every norm bounded increasing net $(x_i)_{i \in I}$ (see [24, Definition 2.4.18]). Recall the following two facts [24, Proposition 2.4.19]:

- Every dual Banach lattice is monotonically complete.
- If X is monotonically complete, then it has the weak Fatou property, i.e., there exists an r only depending on X such that

$$\left\| \sup_{i \in I} x_i \right\| \leq r \sup_{i \in I} \|x_i\|.$$

If X is a monotonically complete HL space, then the Hardy-Littlewood maximal function

$$(\widetilde{M}f)(\xi) := \sup_{r>0} \frac{1}{|B_\xi(r)|} \int_{B_\xi(r)} |f(\eta)| d\eta, \quad \xi \in \mathbb{R}^d,$$

is well-defined and bounded on each $L^p(\mathbb{R}^d; X)$.

It is known (see [12, Theorem 2.8]) that HL spaces are p -convex for some $p \in (1, \infty)$, i.e., there is a constant C such that

$$\left\| \left(\sum_{n=1}^N |x_n|^p \right)^{1/p} \right\| \leq C_p \left(\sum_{n=1}^N \|x_n\|^p \right)^{1/p}$$

for all finite subsets x_1, \dots, x_N in X . It is easy to check that $X = L^\infty$ has the HL property. In [12, Proposition 2.4, Remark 2.9] it is shown that ℓ^1 fails the HL property.

The following deep result is proved in [3] and [39, Theorem 3].

Proposition 4.2. *For a Banach function space X the following assertions are equivalent:*

- (1) X is a UMD space;
- (2) X and X^* are HL spaces.

We will be interested in the ℓ^s -boundedness of the family of convolution operators whose kernels $k \in L^1(\mathbb{R}^d)$ satisfy the almost everywhere pointwise bound

$$(4.1) \quad |k * f| \leq Mf$$

for all simple $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Let us denote by $\widetilde{\mathcal{K}}$ the set of all such kernels.

Lemma 4.3. *For every $k \in \widetilde{\mathcal{K}}$ one has $\|k\|_{L^1(\mathbb{R}^d)} \leq 1$.*

Proof. From

$$|k * f(x) - k * f(x')| = \left| \int_{\mathbb{R}} [k(x-y) - k(x'-y)] f(y) dy \right| \leq \|k(x-\cdot) - k(x'-\cdot)\|_1 \|f\|_\infty$$

and the L^1 -continuity of translations it follows that $k * f$ is a continuous function for $k \in L^1$ and $f \in L^\infty(\mathbb{R}^d)$. For all functions $f \in L^\infty(\mathbb{R}^d)$ with $|f| \leq 1$ almost everywhere, it follows from the assumption on k and the observation just made that for all $x \in \mathbb{R}^d$,

$$(4.2) \quad |k * f(x)| \leq \widetilde{M}f(x) \leq 1.$$

Consider the functions $f_n(y) := \text{sign}(k(-y))$. Then (4.2) implies that

$$\int_{[-n,n]^d} |k(-y)| dy = |k * f_n(0)| \leq 1.$$

Letting n tend to infinity, we find that $\|k\|_{L^1(\mathbb{R}^d)} \leq 1$. □

Below we present classes of examples of such kernels. In particular, if a kernel k is radially decreasing, then $k \in \widetilde{\mathcal{K}}$ if and only if $\|k\|_{L^1(\mathbb{R}^d)} \leq 1$ (see Proposition 4.5 below).

The next proposition shows that in (4.1) we may replace the range space \mathbb{R} by an arbitrary Banach lattice. Of course, this result is trivial in the case of Banach function spaces, where the estimate holds in a pointwise sense.

Proposition 4.4. *Let $k \in \widetilde{\mathcal{K}}$ and let X be a monotonically complete Banach lattice. If $f : \mathbb{R}^d \rightarrow X$ is a simple function, then almost everywhere*

$$|k * f| \leq \widetilde{M}|f|.$$

Proof. For all $0 \leq x^* \in E^*$ and $\xi \in \mathbb{R}^d$ we have (see [24, Proposition 1.3.7, Lemma 1.4.4])

$$\begin{aligned} \langle |k * f(\xi)|, x^* \rangle &= \sup_{|y^*| \leq x^*} \langle k * f(\xi), y^* \rangle \\ &= \sup_{|y^*| \leq x^*} k * \langle f, y^* \rangle(\xi) \\ &\leq \sup_{|y^*| \leq x^*} (\widetilde{M}\langle f, y^* \rangle)(\xi) \\ &\leq \sup_{|y^*| \leq x^*} \langle (\widetilde{M}f)(\xi), |y^*| \rangle = \langle (\widetilde{M}f)(\xi), x^* \rangle \end{aligned}$$

and the result follows [24, Proposition 1.4.2]. \square

The next result is well known in the scalar-valued case (see [15, Chapter 2]). The concise proof presented here was kindly shown to us by Tuomas Hytönen.

Proposition 4.5. *Let X be a monotonically complete Banach lattice. If $k : \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable function satisfying*

$$\int_{\mathbb{R}^d} \text{ess sup}_{|\eta| \geq |\xi|} |k(\eta)| d\xi \leq 1,$$

then $\|k\|_{L^1(\mathbb{R}^d)} \leq 1$ and for all $f \in L^p(\mathbb{R}^d; X)$, $1 \leq p \leq \infty$, we have the pointwise estimate

$$|k * f| \leq \widetilde{M}f.$$

Proof. By Proposition 4.4 it suffices to consider the case $X = \mathbb{R}$. Put $h(r) := \text{ess sup}_{|\xi| \geq r} |k(\xi)|$. Then h is non-increasing, right-continuous and vanishes at infinity; hence

$$h(r) = \int_{(r, \infty)} d\mu(t)$$

for a positive measure μ on $\mathbb{R}_+ = (0, \infty)$. Thus

$$\begin{aligned} |k * f(\xi)| &= \left| \int_{\mathbb{R}^d} k(\eta) f(\xi - \eta) d\eta \right| \leq \int_{\mathbb{R}^d} h(|\eta|) |f(\xi - \eta)| d\eta \\ &= \int_{\mathbb{R}^d} \int_{(|\eta|, \infty)} |f(\xi - \eta)| d\mu(t) d\eta \\ &= \int_{\mathbb{R}_+} \int_{B(0, t)} |f(\xi - \eta)| d\eta d\mu(t) \leq \int_{\mathbb{R}_+} |B(0, t)| \widetilde{M}f(\xi) d\mu(t). \end{aligned}$$

Further, writing $S(0, r)$ for the sphere in \mathbb{R}^d of radius r centered at the origin and $|S(0, r)|_{d-1}$ for its $(d-1)$ -dimensional measure, it follows by using polar coordinates that

$$\begin{aligned} \int_{\mathbb{R}_+} |B(0, t)| d\mu(t) &= \int_{\mathbb{R}_+} \int_0^t |S(0, r)|_{d-1} dr d\mu(t) \\ &= \int_0^\infty \int_{(r, \infty)} |S(0, r)|_{d-1} d\mu(t) dr \end{aligned}$$

$$= \int_0^\infty h(r) |S(0, r)|_{d-1} dr = \int_{\mathbb{R}^d} \operatorname{ess\,sup}_{|\eta| \geq |\xi|} |k(\eta)| d\xi \leq 1.$$

Hence $|k * f(\xi)| \leq \widetilde{M}|f|(\xi)$. \square

The next result shows that the above sufficient condition holds under a certain integrability condition on the derivative:

Proposition 4.6. *If $k \in W_{\text{loc}}^{1,1}(\mathbb{R}^d \setminus \{0\})$ satisfies $\lim_{|\xi| \rightarrow \infty} k(\xi) = 0$, then*

$$\int_0^\infty \operatorname{ess\,sup}_{|\eta| \geq |\xi|} |k(\eta)| d\xi \lesssim_d \int_0^\infty \rho^d \operatorname{ess\,sup}_{\xi \in S} |\nabla k(\rho\xi)| d\rho,$$

where $S = S(0, 1)$ is the unit sphere in \mathbb{R}^d . In particular, if the right-hand side is finite and X is a monotonically complete Banach lattice, then for all $f \in L^p(\mathbb{R}^d; X)$, $1 \leq p \leq \infty$, almost everywhere we have the pointwise estimate

$$|k * f| \lesssim_d \left(\int_0^\infty \rho^d \sup_{\xi \in S} |\nabla k(\rho\xi)| d\rho \right) \widetilde{M}|f|.$$

Proof. Using polar coordinates,

$$\begin{aligned} \int_{\mathbb{R}^d} \operatorname{ess\,sup}_{|\eta| \geq |\xi|} |k(\eta)| d\xi &= \int_{\mathbb{R}^d} \operatorname{ess\,sup}_{\eta \in S} \sup_{r \geq |\xi|} |k(r\eta)| d\xi \\ &\leq \int_{\mathbb{R}^d} \operatorname{ess\,sup}_{\eta \in S} \sup_{r \geq |\xi|} \int_r^\infty |\nabla k(\rho\eta)| d\rho d\xi \\ &= \int_{\mathbb{R}^d} \operatorname{ess\,sup}_{\eta \in S} \sup_{r \geq |\xi|} \int_0^\infty \mathbf{1}_{(0, \rho)}(r) |\nabla k(\rho\eta)| d\rho d\xi \\ &\leq \int_0^\infty \int_{\mathbb{R}^d} \operatorname{ess\,sup}_{\eta \in S} \sup_{r \geq |\xi|} \mathbf{1}_{(0, \rho)}(r) |\nabla k(\rho\eta)| d\xi d\rho \\ &\simeq_d \int_0^\infty \rho^d \operatorname{ess\,sup}_{\eta \in S} |\nabla k(\rho\eta)| d\rho. \end{aligned}$$

Therefore, the result follows from Proposition 4.5. \square

Recall the definition

$$\widetilde{\mathcal{K}} := \{k \in L^1(\mathbb{R}^d) : |k * f| \leq \widetilde{M}|f| \text{ a.e. for all simple } f : \mathbb{R}^d \rightarrow \mathbb{R}\}.$$

For a kernel $k \in L^1(\mathbb{R}^d)$ we denote by T_k the associated convolution operator $f \mapsto k * f$ on $L^p(\mathbb{R}^d; X)$.

If X is a UMD Banach function space and $s \in (1, \infty)$, then $X(\ell^s)$ is a UMD Banach function space again (see [39, p. 214]). This implies that the family $\{T_k : k \in \widetilde{\mathcal{K}}\}$ is ℓ^s -bounded. Indeed, using (3.2), for all finite sequences $(k_n)_{n=1}^N$ in $\widetilde{\mathcal{K}}$ and $(f_n)_{n=1}^N$ in $X(\ell_N^s)$ we have

$$\begin{aligned} \|(k_n * f_n)_{n=1}^N\|_{L^p(\mathbb{R}^d; X(\ell_N^s))} &\leq \|(\widetilde{M}f_n)_{n=1}^N\|_{L^p(\mathbb{R}^d; X(\ell_N^s))} \\ &\leq C_{X,s} \|(f_n)_{n=1}^N\|_{L^p(\mathbb{R}^d; X(\ell_N^s))}, \end{aligned}$$

where we applied Proposition 4.2 to $X(\ell_N^s)$. A similar but simpler argument give that this result extends to $s = \infty$.

For $s = 1$ this argument does not work since the maximal function is not bounded on ℓ^1 . Surprisingly, we can still obtain the following result for $s = 1$, which is the main result of this section.

Theorem 4.7. *Let X be a Banach lattice, let $p \in (1, \infty)$, and consider the family of convolution operators $\widetilde{\mathcal{T}} = \{T_k : k \in \mathcal{K}\}$ on $L^p(\mathbb{R}^d; X)$.*

- (1) *If X^* is an HL lattice, then $\widetilde{\mathcal{T}}$ is ℓ^1 -bounded on $L^p(\mathbb{R}^d; X)$.*
- (2) *If X is a UMD Banach function space, then $\widetilde{\mathcal{T}}$ is ℓ^s -bounded on $L^p(\mathbb{R}^d; X)$ for all $s \in [1, \infty]$.*

Remark 4.8. It is crucial that the case $s = 1$ is included here, i.e., the set \mathcal{T} is ℓ^1 -bounded on each $L^p(\mathbb{R}; X)$. This fact will be needed in the proof of our main result about R -boundedness of stochastic convolution operators (Theorem 7.2 below).

Before turning to the proof of the theorem we start with some preparations and motivating results. The next proposition shows that in the case of Banach function spaces, in a certain sense ℓ^s -boundedness of operator families becomes more restrictive as s decreases.

Proposition 4.9. *Let X be a Banach function space. Let $1 \leq s < t < \infty$ and $p \in (1, \infty)$ and $q = pt/s$. Let $\mathcal{T}_+ = \{T_k : k \in \mathcal{K}, k \geq 0\}$. If \mathcal{T}_+ is ℓ^s -bounded on $L^p(\mathbb{R}^d; X)$, then \mathcal{T}_+ is ℓ^t -bounded on $L^q(\mathbb{R}^d; X)$.*

Proof. Let $0 \leq k_1, \dots, k_N \in \mathcal{K}$ be non-negative kernels and let $f_1, \dots, f_N : \mathbb{R}^d \rightarrow X$ be simple functions. By Lemma 4.3 we have $\|k_n\|_{L^1(\mathbb{R}^d)} \leq 1$, and hence Jensen's inequality implies $|k_n * f_n|^t \leq |k_n * (|f_n|^{t/s})|^s$. Therefore,

$$\begin{aligned} \|(|k_n * f_n|)_{n=1}^N\|_{L^q(X(\ell^t))} &\leq \|(|k_n * (|f_n|^{t/s})|)_{n=1}^N\|_{L^p(X(\ell^s))}^{s/t} \\ &\leq C \|(|f_n|^{t/s})_{n=1}^N\|_{L^p(X(\ell^s))}^{s/t} = C \|(f_n)_{n=1}^N\|_{L^q(X(\ell^t))}. \end{aligned}$$

□

The next proposition gives necessary and sufficient conditions for ℓ^∞ -boundedness in terms of L^p -boundedness of the maximal function \widetilde{M} .

Proposition 4.10. *Let X be a Banach lattice, let $p \in [1, \infty]$ and consider the family of convolution operators $\widetilde{\mathcal{T}} = \{T_k : k \in \mathcal{K}\}$ on $L^p(\mathbb{R}^d; X)$.*

- (1) *If $\widetilde{\mathcal{T}}$ is ℓ^∞ -bounded on $L^p(\mathbb{R}^d; X)$, then \widetilde{M} is $L^p(\mathbb{R}^d; X)$ -bounded;*
- (2) *If \widetilde{M} is $L^p(\mathbb{R}^d; X)$ -bounded and X is monotonically complete, then $\widetilde{\mathcal{T}}$ is ℓ^∞ -bounded on $L^p(\mathbb{R}^d; X)$.*

Although the proof below also works for $p = 1$, the maximal function is of course not bounded on $L^1(\mathbb{R}^d; X)$ (see [15]). As a consequence we see that $\widetilde{\mathcal{T}}$ is not ℓ^∞ -bounded on $L^1(\mathbb{R}; X)$.

Proof. (1): For all $r > 0$ and simple $f : \mathbb{R}^d \rightarrow \mathbb{R}$ we have

$$\begin{aligned} \frac{1}{|B_0(r)|} |\mathbf{1}_{B_0(r)} * f(\xi)| &= \frac{1}{|B_0(r)|} \left| \int_{\mathbb{R}^d} \mathbf{1}_{B_0(r)}(\xi - \eta) f(\eta) d\eta \right| \\ &= \frac{1}{|B_\xi(r)|} \left| \int_{\mathbb{R}^d} \mathbf{1}_{B_\xi(r)}(\eta) f(\eta) d\eta \right|. \end{aligned}$$

It follows that the functions $k_r := \frac{1}{|B_0(r)|} \mathbf{1}_{B_0(r)}$ belong to \mathcal{K} for all $r > 0$. Moreover, the above identities extend to functions $f \in L^p(\mathbb{R}^d; X)$ provided we interpret $|\cdot|$

as the modulus in X . As a consequence, for all $f \in L^p(\mathbb{R}^d; X)$ and all finite sets $J \subseteq \mathbb{R}_+$, the ℓ^∞ -boundedness of \mathcal{T} on $L^p(\mathbb{R}^d; X)$ implies

$$\|\widetilde{M}_J f\|_{L^p(\mathbb{R}^d; X)} = \left\| \sup_{r \in J} |T_{k_r} f| \right\|_{L^p(\mathbb{R}^d; X)} \lesssim_{p,d,X} \|f\|_{L^p(\mathbb{R}^d; X)}.$$

It follows that the mappings \widetilde{M}_J are bounded on $L^p(\mathbb{R}^d; X)$, uniformly with respect to J .

(2): Let $k_1, \dots, k_N \in \widetilde{\mathcal{K}}$ and simple $f_1, \dots, f_N \in L^p(\mathbb{R}^d; X)$ be given. Then, by Proposition 4.4,

$$\begin{aligned} \int_{\mathbb{R}^d} \left\| \sup_{1 \leq n \leq N} |T_{k_n} f_n(\xi)| \right\|^p d\xi &\leq \int_{\mathbb{R}^d} \left\| \sup_{1 \leq n \leq N} (\widetilde{M}|f_n|)(\xi) \right\|^p d\xi \\ &\leq \int_{\mathbb{R}^d} \left\| \widetilde{M} \left(\sup_{1 \leq n \leq N} |f_n| \right) \right\|^p d\xi \\ &\lesssim_{p,d,X} \int_{\mathbb{R}^d} \left\| \sup_{1 \leq n \leq N} |f_n(\xi)| \right\|^p d\xi, \end{aligned}$$

with the obvious modifications for $p = \infty$. By approximation, this estimate extends to general $f_1, \dots, f_N \in L^p(\mathbb{R}^d; X)$. \square

Proof of Theorem 4.7. Fix $1 < p < \infty$. We begin by observing that for $k \in L^1(\mathbb{R}^d)$ the adjoint of T_k as an operator on $L^p(\mathbb{R}^d)$ equals $T_{\bar{k}}$ as an operator on $L^{p'}(\mathbb{R}^d)$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $\bar{k}(x) = k(-x)$. Clearly, $T_{\bar{k}} \in \mathcal{T}$.

By definition (if X^* is HL), respectively by Proposition 4.2 (if X is UMD), \widetilde{M} is $L^{p'}(\mathbb{R}^d; X^*)$ -bounded. Therefore, by Proposition 4.10, \mathcal{T} is ℓ^∞ -bounded on $L^{p'}(\mathbb{R}^d; X^*)$, and Proposition 3.4(2) then shows that \mathcal{T} is ℓ^1 -bounded on $L^p(\mathbb{R}^d; X)$.

If X is UMD, we have already sketched a proof in the case $s \in (1, \infty]$ (alternatively we can use interpolation). We may apply the above argument to p' and X^* as well and obtain that \mathcal{T} is also ℓ^∞ -bounded on $L^p(\mathbb{R}^d; X)$. Now the result follows from Proposition 3.4(3). Here we used that a UMD space X is reflexive and thus $L^p(\mathbb{R}^d; X)$ is reflexive (see [10]) and hence has order continuous norm. \square

Remark 4.11. If a family of kernels $\mathcal{K} = \{k : \mathbb{R}^d \rightarrow \mathbb{R}\}$ satisfies an appropriate smoothness condition, then the ℓ^s -boundedness of $\{T_k : k \in \mathcal{K}\}$ as a family of operators on $L^{p_0}(\mathbb{R}^d; X)$ for a certain $p_0 \in [1, \infty]$ implies the ℓ^s -boundedness on $L^p(\mathbb{R}^d; X)$ for all $p \in (1, \infty)$ (see [14, Theorem V.3.4]). This result is interesting from a theoretical point of view, but in all applications considered here we can consider arbitrary $p \in (1, \infty)$ from the beginning without additional difficulty. The main reason for this is the p -independence of the HL property.

The next example shows that Theorem 4.7 does not extend to $p = 1$.

Example 4.12. Let $X = \ell^r$ with $r \in (1, \infty)$ fixed. By Theorem 4.7, the family \mathcal{T}_1 considered there is ℓ^s -bounded on $L^p(\mathbb{R}; \ell^r)$ for all $p \in (1, \infty)$ and $s \in [1, \infty]$. We show that it fails to be ℓ^s -bounded on $L^1(\mathbb{R}; \ell^r)$ for all $s \in [1, \infty]$.

Let $\lambda_n > 0$ with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Let $k_n(t) = \frac{1}{2} \lambda_n e^{-\lambda_n |t|}$. By Proposition 4.5 we have $(k_n)_{n \geq 1} \subseteq \widetilde{\mathcal{K}}$ and $\{T_{k_n} : n \geq 1\} \subseteq \widetilde{\mathcal{T}}$. The kernels $(k_n^2)_{n \geq 1}$, are precisely the ones which are needed in [29, Section 7].

Fix $s \in [1, \infty]$. We will show that $\{T_{k_n} : n \geq 1\}$ is not ℓ^s -bounded as a family of operators on $L^1(\mathbb{R}; \ell^r)$. Indeed, assume it is ℓ^s -bounded on this space with

constant C . Then, letting $N \rightarrow \infty$ in the definition of ℓ^s -boundedness and using the identification $(L^1(\mathbb{R}; \ell^r))(\ell^s) = L^1(\mathbb{R}; \ell^r(\ell^s))$ (see (3.3)), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \left(\sum_{j \geq 1} \left[\left(\sum_{n \geq 1} \left| \int_{\mathbb{R}} k_n(u) f_{nj}(t-u) du \right|^s \right)^{1/s} \right]^r \right)^{1/r} dt \\ \leq C \int_{\mathbb{R}} \left(\sum_{j \geq 1} \left[\left(\sum_{n \geq 1} |f_{nj}(t)|^s \right)^{1/s} \right]^r \right)^{1/r} dt, \end{aligned}$$

where $(f_{nj})_{n,j \geq 1}$ is in $L^1(\mathbb{R}; \ell^r(\ell^s))$; we make the obvious modifications if $s = \infty$. Taking $f_{nj}(t) = f_j(t) \delta_{nj}$ yields

$$\int_{\mathbb{R}} \left(\sum_{j \geq 1} \left| \int_{\mathbb{R}} k_j(u) f_j(t-u) du \right|^r \right)^{1/r} dt \leq C \int_{\mathbb{R}} \left(\sum_{j \geq 1} |f_j(t)|^r \right)^{1/r} dt.$$

The latter is easily seen to be equivalent to the maximal L^1 -regularity of the diagonal operator $Ae_j = \lambda_j e_j$ on ℓ^r , which does not hold by [16].

5. THE OPERATORS N_k

Let X be a Banach space and H be a Hilbert space. For $k \in L^2(\mathbb{R}^d)$ and simple functions $G : \mathbb{R}^d \rightarrow H \otimes X$ we define the function $N_k G : \mathbb{R}^d \rightarrow L^2(\mathbb{R}^d) \otimes H \otimes X$ by

$$(5.1) \quad ((N_k G)(t))(s) = k(t-s)G(s), \quad s, t \in \mathbb{R}^d.$$

Proposition 5.1. *Let X be a non-zero Banach space, H a non-zero Hilbert space, and let $p \in [1, \infty)$ and $d \geq 1$ be arbitrary and fixed. The following assertions are equivalent:*

- (1) X has type 2 and $p \in [2, \infty)$;
- (2) For all $k \in L^2(\mathbb{R}^d)$ the operator $G \mapsto N_k G$ extends to a bounded operator

$$N_k : L^p(\mathbb{R}^d; \gamma(H, X)) \rightarrow L^p(\mathbb{R}^d, dt; \gamma(L^2(\mathbb{R}^d; H, ds), X)).$$

Proof. (1) \Rightarrow (2): It suffices to prove that for any Banach space Y with type 2 the mapping $G \mapsto N_k G$ extends to a bounded operator

$$N_k : L^p(\mathbb{R}^d; Y) \rightarrow L^p(\mathbb{R}^d, dt; L^2(\mathbb{R}^d, ds; Y)).$$

Indeed, once this has been shown we take $Y = \gamma(H, X)$ (which has type 2 if X has type 2) and apply Proposition 2.4(1).

Fix $p \geq 2$ and $f \in L^p(\mathbb{R}^d; Y)$. By Young's inequality,

$$\begin{aligned} \|N_k f\|_{L^p(\mathbb{R}^d, dt; L^2(\mathbb{R}^d, ds; Y))}^p &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |k(t-s)|^2 \|f(s)\|^2 ds \right)^{p/2} dt \\ &= \| |k|^2 * \|f\|^2 \|_{L^{p/2}(\mathbb{R}^d)}^{p/2} \\ &\leq \|k^2\|_{L^1(\mathbb{R}^d)}^{p/2} \| \|f\|^2 \|_{L^{p/2}(\mathbb{R}^d)}^{p/2} = \|k\|_{L^2(\mathbb{R}^d)}^p \|f\|_{L^p(\mathbb{R}^d; Y)}^p. \end{aligned}$$

(2) \Rightarrow (1): To show that $p \in [2, \infty)$ it suffices to argue on one-dimensional subspaces of X . We may therefore assume that $X = H = \mathbb{R}$ and therefore $\gamma(L^2(\mathbb{R}^d); X) = L^2(\mathbb{R}^d)$.

Let $k = \mathbf{1}_{(a,b)^d}$ with $a < b$ and set $\delta := b - a$. For $0 < r < \delta/2$ let $G_r := \mathbf{1}_{(0,r)^d}$. Then

$$\|N_k G_r\|_{L^p(\mathbb{R}^d, dt; L^2(\mathbb{R}^d, ds))}^p = \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} k^2(s) G_r^2(t-s) ds \right)^{p/2} dt$$

$$\geq \int_{((a+b)/2, b)^d} \left(\int_{t-(0, r)^d} 1 \, ds \right)^{p/2} dt = r^{dp/2} (\delta/2)^d.$$

On the other hand, $\|G_r\|_{L^p(\mathbb{R}^d)}^p = r^d$. Therefore, $\|N_k\| \geq r^{d(\frac{1}{2} - \frac{1}{p})} (\delta/2)^d$. Letting $r \downarrow 0$, the boundedness of N_k forces that $p \in [2, \infty)$.

To show that X has type 2 we may assume that $H = \mathbb{R}$ (identify X with a closed subspace of $\gamma(H, X)$ via the mapping $x \mapsto h_0 \otimes x$, where $h_0 \in H$ is some fixed norm one vector).

As before let $k = \mathbf{1}_{(a, b)^d}$ with $a < b$ and fix $0 < r \leq \delta/2$ with $\delta = b - a$. Fix a simple function $G : \mathbb{R}^d \rightarrow X$ with support in $I = (0, r)^d$. For all $t \in J = ((a+b)/2, b)^d$, one has

$$\|G\|_{\gamma(L^2(I), X)} = \|k(t - \cdot)G\|_{\gamma(L^2(I), X)}.$$

It follows that

$$\begin{aligned} |J|^{1/p} \|G\|_{\gamma(L^2(I), X)} &= \|t \mapsto k(t - \cdot)G\|_{L^p(J; \gamma(L^2(I), X))} \\ &\leq \|t \mapsto k(t - \cdot)G\|_{L^p(\mathbb{R}^d; \gamma(L^2(\mathbb{R}^d), X))} \\ &= \|N_k G\|_{L^p(\mathbb{R}^d; \gamma(L^2(\mathbb{R}^d), X))} \\ &\leq \|N_k\| \|G\|_{L^p(\mathbb{R}^d; X)} = \|N_k\| \|G\|_{L^p(I; X)}. \end{aligned}$$

As a consequence, the identity mapping on $L^p(I) \otimes X$ extends to a bounded operator from $L^p(I; X)$ to $\gamma(L^2(I), X)$. Hence by [38, Proposition 6.1], X has type 2. \square

Inspection of the proof shows that the following weaker version of (2) already implies (1):

(2') *There exist real numbers $a < b$ such that the mapping $G \mapsto N_{\mathbf{1}_{(a, b)^d}} G$ extends to a bounded operator*

$$N_{\mathbf{1}_{(a, b)^d}} : L^p(\mathbb{R}^d; X) \rightarrow L^p(\mathbb{R}^d; \gamma(L^2(\mathbb{R}^d), X)).$$

In view of this we shall assume from now on that X has type 2 and consider only exponents $p \in [2, \infty)$. We now fix a subset $\mathcal{K} \subseteq L^2(\mathbb{R}^d)$ and consider the family

$$\mathcal{N}_{\mathcal{K}} := \{N_k : k \in \mathcal{K}\}.$$

By the previous result, the operators in $\mathcal{N}_{\mathcal{K}}$ extend to bounded operators from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; \gamma(L^2(\mathbb{R}^d), X))$. By slight abuse of notation, the resulting family of extensions will be denoted by $\mathcal{N}_{\mathcal{K}}$ again.

In the next result we investigate the role of H with regard to the R -boundedness properties of $\mathcal{N}_{\mathcal{K}}$.

Proposition 5.2 (Independence of H). *Let X be a Banach space with type 2, H be a non-zero Hilbert space, and $p \in [2, \infty)$. For any set $\mathcal{K} \subseteq L^2(\mathbb{R}^d)$, the following assertions are equivalent:*

- (1) *The family $\mathcal{N}_{\mathcal{K}}$ is R -bounded from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; \gamma(L^2(\mathbb{R}^d), X))$;*
- (2) *The family $\mathcal{N}_{\mathcal{K}}$ is R -bounded from $L^p(\mathbb{R}^d; \gamma(H, X))$ to $L^p(\mathbb{R}^d; \gamma(L^2(\mathbb{R}^d; H), X))$.*

Proof. We only need to prove that (1) implies (2); the converse implication follows by restricting to a one-dimensional subspace of H and identifying $\gamma(\mathbb{R}, X)$ with X .

Suppose now that (1) holds. By Proposition 2.3 each operator in $\mathcal{N}_{\mathcal{K}}$ extends to a bounded operator from $\gamma(H, L^p(\mathbb{R}^d; X))$ to $\gamma(H, L^p(\mathbb{R}^d; \gamma(L^2(\mathbb{R}^d), X)))$ and the resulting family of extensions is again R -bounded. By the γ -Fubini isomorphism (Proposition 2.2), $\mathcal{N}_{\mathcal{K}}$ extends to an R -bounded family of operators from

$L^p(\mathbb{R}^d; \gamma(H, X))$ to $L^p(\mathbb{R}^d; \gamma(H, \gamma(L^2(\mathbb{R}^d), X)))$. Now the result follows from the fact that $\gamma(H, \gamma(L^2(\mathbb{R}^d), X))$ embeds continuously into $\gamma(L^2(\mathbb{R}^d; H), X)$ by Propositions 2.6 and 2.7. \square

The main result of this section reduces the problem of proving R -boundedness of a family of operators N_k to proving ℓ^1 -boundedness of the corresponding family of convolution operators T_{k^2} (see Section 4 for the definition of these operators).

We recall from Proposition 2.9 and its proof that a Banach lattice has type 2 if and only if it has martingale type 2, and that such a Banach lattice is 2-convex. Because of this, its 2-concavification X^2 is a Banach lattice again. If X is a Banach function space over some measure space (S, μ) (this is the only case we shall consider), X^2 consists of all measurable functions $f : S \rightarrow \mathbb{R}$ such that $|f| = g^2$ for some $g \in X$, identifying functions which are equal μ -almost everywhere. For example, when $X = L^q(S)$ with $q \in [2, \infty)$, then $X^2 = L^{q/2}(S)$.

Theorem 5.3. *Let X be a Banach lattice with type 2 and let X^2 denote its 2-concavification. Let $p \in [2, \infty)$. For any set of kernels $\mathcal{K} \subseteq L^2(\mathbb{R}^d)$, the following assertions are equivalent:*

(1) *The family*

$$\mathcal{N}_{\mathcal{K}} := \{N_k : k \in \mathcal{K}\}$$

is R -bounded from $L^p(\mathbb{R}^d; X)$ to $L^p(\mathbb{R}^d; \gamma(L^2(\mathbb{R}^d), X))$;

(2) *The family*

$$\mathcal{T}_{\mathcal{K}^2} := \{T_{k^2} : k \in \mathcal{K}\}$$

is ℓ^1 -bounded on $L^{p/2}(\mathbb{R}^d; X^2)$.

Moreover, $R(\mathcal{N}_{\mathcal{K}}) \approx_{p,X} (R^1(\mathcal{T}_{\mathcal{K}^2}))^{1/2}$.

Proof. (2) \Rightarrow (1): Assume that $\mathcal{T}_{\mathcal{K}^2}$ is ℓ^1 -bounded and fix $k_1, \dots, k_N \in \mathcal{K}$. Let $G_1, \dots, G_N \in L^p(\mathbb{R}^d; X)$ be simple functions. As X has type 2, it also has finite cotype (see [1, Example 11.1.2 and Theorem 11.1.14]). Since each G_n takes values in a finite-dimensional subspace of X , a standard argument shows that we may assume X is a Banach function space (see [41, Theorem 3.9]).

Set $f_n := |G_n|^2$. By Lemma 2.8,

$$\begin{aligned} \mathbb{E}_r \left\| \sum_{n=1}^N r_n (N_{k_n} G_n)(t) \right\|_{\gamma(L^2(\mathbb{R}^d), X)}^p &= \mathbb{E}_r \left\| s \mapsto \sum_{n=1}^N r_n k_n(t-s) G_n(s) \right\|_{\gamma(L^2(\mathbb{R}^d), X)}^p \\ &\approx_{p,X} \left\| \left(\sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(t-s) G_n(s)|^2 ds \right)^{1/2} \right\|^p \\ &= \left\| \left(\sum_{n=1}^N \int_{\mathbb{R}^d} k_n^2(t-s) f_n(s) ds \right)^{1/2} \right\|^p \\ &= \left\| \left(\sum_{n=1}^N k_n^2 * f_n(t) \right)^{1/2} \right\|^p \\ &= \left\| \sum_{n=1}^N k_n^2 * f_n(t) \right\|_{X^2}^{p/2}. \end{aligned}$$

Integrating over \mathbb{R}^d , it follows that

$$\begin{aligned} \mathbb{E}_r \left\| \sum_{n=1}^N r_n (N_{k_n} G_n) \right\|_{L^p(\mathbb{R}^d; \gamma(L^2(\mathbb{R}^d), X))}^p &= \int_{\mathbb{R}^d} \left\| \sum_{n=1}^N k_n^2 * f_n \right\|_{X^2}^{p/2} dt \\ &= \left\| \sum_{n=1}^N k_n^2 * f_n \right\|_{L^{p/2}(\mathbb{R}^d; X^2)}^{p/2} \\ &\leq (R^1(\mathcal{T}_{\mathcal{H}^2}))^{p/2} \left\| \sum_{n=1}^N f_n \right\|_{L^{p/2}(\mathbb{R}^d; X^2)}^{p/2}. \end{aligned}$$

Now for the latter one has

$$\begin{aligned} \left\| \sum_{n=1}^N f_n \right\|_{L^{p/2}(\mathbb{R}^d; X^2)}^{p/2} &= \left\| \left(\sum_{n=1}^N |G_n|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d; X)}^p \\ &\approx_X \left\| \sum_{n=1}^N r_n G_n \right\|_{L^p(\mathbb{R}^d; L^2(\Omega_r; X))}^p \leq \mathbb{E}_r \left\| \sum_{n=1}^N r_n G_n \right\|_{L^p(\mathbb{R}^d; X)}^p. \end{aligned}$$

Combing the estimates, the result follows.

(1) \Rightarrow (2): This is proved similarly. \square

6. STOCHASTIC INTEGRATION

We begin recalling some basic facts from the theory of stochastic integration in UMD Banach spaces as developed in [26] (for a survey see [29]).

Let (Ω, \mathbb{P}) be a probability space and let H be a Hilbert space. An *H-cylindrical Brownian motion* is a bounded linear operator W_H from $L^2(\mathbb{R}_+; H)$ to $L^2(\Omega)$ such that

- (i) for all $f \in L^2(\mathbb{R}_+; H)$ the random variable $W_H f$ is centered Gaussian;
- (ii) for all $f, g \in L^2(\mathbb{R}_+; H)$ we have $\mathbb{E}(W_H f \cdot W_H g) = [f, g]_{L^2(\mathbb{R}_+; H)}$.

If (Ω, \mathbb{P}) is endowed with a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$, we call W_H a *H-cylindrical \mathcal{F} -Brownian motion* on H if $W_H f$ is independent of \mathcal{F}_t for all $f \in L^2(\mathbb{R}_+; H)$ with support in (t, ∞) . In that case, $t \mapsto W_H(\mathbf{1}_{(0,t)} \otimes h)$ is an \mathcal{F} -Brownian motion for all $h \in H$, which is standard if $\|h\| = 1$. Two such Brownian motions are independent if and only if the corresponding vectors h are orthogonal. If there is no danger of confusion we also use the standard notation $W_H(t)h$ for the random variable $W_H(\mathbf{1}_{(0,t)} \otimes h)$.

For $0 \leq a < b < \infty$, $x \in X$, and an \mathcal{F}_a -measurable set $A \subseteq \Omega$, the stochastic integral of the indicator process $(t, \omega) \mapsto \mathbf{1}_{(a,b] \times A}(t, \omega) h \otimes x$ with respect to W_H is defined as

$$\int_0^t \mathbf{1}_{(a,b] \times A} \otimes (h \otimes x) dW_H := \mathbf{1}_A W_H(\mathbf{1}_{(a \wedge t, b \wedge t]} \otimes h) \otimes x, \quad t \in \mathbb{R}_+.$$

By linearity, this definition extends to *adapted finite rank step processes*, which we define as finite linear combinations of indicator processes of the above form.

Proposition 6.1 (Burkholder inequality for martingale type 2 spaces; see [4, 5, 33]). *Let X have martingale type 2 and let $p \in (1, \infty)$ be fixed. For all adapted*

finite rank step processes G we have

$$\mathbb{E} \sup_{t \in \mathbb{R}_+} \left\| \int_0^t G dW_H \right\|^p \leq C_{p,X}^p \|G\|_{L^p(\Omega; L^2(\mathbb{R}_+; \gamma(H, X)))}^p.$$

Under the identification

$$\mathbf{1}_{(a,b] \times A} \otimes (h \otimes x) = \mathbf{1}_A \otimes ((\mathbf{1}_{(a,b]} \otimes h) \otimes x),$$

we may identify finite rank step processes with elements in $L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ and we have the following estimate.

Proposition 6.2. [26, Theorems 5.9, 5.12] *Let X be a UMD Banach space and let $p \in (1, \infty)$ be fixed. For all adapted finite rank step processes G we have*

$$c^p \mathbb{E} \|G\|_{\gamma(L^2(\mathbb{R}_+; H), X)}^p \leq \mathbb{E} \sup_{t \in \mathbb{R}_+} \left\| \int_0^t G dW_H \right\|^p \leq C^p \mathbb{E} \|G\|_{\gamma(L^2(\mathbb{R}_+; H), X)}^p,$$

with constants $0 < c \leq C < \infty$ independent of G .

When G does not depend on Ω the UMD condition can be omitted in the above result.

By a standard density argument (see [26] for details), the stochastic integral has a unique extension to the Banach space $L_{\mathcal{F}}^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ of all adapted elements in $L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$, that is, the closure in $L^p(\Omega; \gamma(L^2(\mathbb{R}_+; H), X))$ of all adapted simple processes with values in $H \otimes X$. In the remainder of this paper, all stochastic integrals are understood in this sense.

6.1. Stochastic convolution operators. For kernels $k \in L^2(\mathbb{R}_+)$ and adapted finite rank step processes $G : \mathbb{R}_+ \times \Omega \rightarrow H \otimes X$ we define the adapted process $S_k G : \mathbb{R}_+ \times \Omega \rightarrow X$ by

$$(6.1) \quad S_k^H G(t) := \int_0^t k(t-s) G(s) dW_H(s), \quad t \in \mathbb{R}_+.$$

Since G is an adapted finite rank step process, the Itô integration theory for scalar-valued processes (see [17, Chapter 17]) shows that the above stochastic integral is well defined for all $t \in \mathbb{R}_+$.

The following observation is a direct consequence of Proposition 6.1 and Young's inequality.

Proposition 6.3. *Let X be a Banach space, H a Hilbert space, and $p \in [2, \infty)$. If X has martingale type 2, the mapping $S_k^H : G \mapsto S_k^H G$ extends to a bounded operator from $L_{\mathcal{F}}^p(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ to $L^p(\mathbb{R}_+ \times \Omega; X)$.*

Note that for deterministic integrands

$$\|S_k^H G(t)\|_{L^p(\Omega; X)} \approx_p \|s \mapsto k(t-s)G(s)\|_{\gamma(L^2(0,t;H), X)}.$$

Therefore, from the proof of Proposition 5.1 we can deduce the following result:

Proposition 6.4. *Let X be a Banach space, H a non-zero Hilbert space, and $p \in [2, \infty)$. The following assertions are equivalent:*

- (1) X has type 2;
- (2) For all $k \in L^2(\mathbb{R}_+)$ the mapping $S_k : G \mapsto S_k G$ extends to a bounded operator from $L_{\mathcal{F}}^p(\mathbb{R}_+; X)$ into $L^p(\mathbb{R}_+ \times \Omega; X)$;
- (3) For all $k \in L^2(\mathbb{R}_+)$ the mapping $S_k^H : G \mapsto S_k^H G$ extends to a bounded operator from $L_{\mathcal{F}}^p(\mathbb{R}_+; \gamma(H, X))$ into $L^p(\mathbb{R}_+ \times \Omega; X)$.

7. R -BOUNDEDNESS OF STOCHASTIC CONVOLUTION OPERATORS

We shall now apply the results of Section 4 to obtain R -boundedness results for stochastic convolution operators. More specifically, we shall provide a connection between R -boundedness of stochastic convolutions with kernel k and ℓ^1 -boundedness of convolutions with the squared kernel k^2 . For $d = 1$, the results of the previous section imply their counterparts for \mathbb{R}_+ by considering functions and kernels supported on \mathbb{R}_+ .

Recall that for $k \in L^2(\mathbb{R}_+)$ the stochastic convolution operators S_k have been defined by (6.1). For a subset $\mathcal{K} \subseteq L^2(\mathbb{R}_+)$ we write $\mathcal{S}_{\mathcal{K}} := \{S_k : k \in \mathcal{K}\}$; we use the same notation for the vector-valued extensions. We will be interested in the R -boundedness of such families. The first result asserts that it suffices to check R -boundedness on deterministic integrands:

Theorem 7.1. *Let X be a Banach space with type 2, H be a non-zero Hilbert space, and let $p \in [2, \infty)$. For a set $\mathcal{K} \subseteq L^2(\mathbb{R}_+)$ the following assertions are equivalent:*

- (1) *The family $\mathcal{S}_{\mathcal{K}}$ is R -bounded from $L^p(\mathbb{R}_+; X)$ to $L^p(\mathbb{R}_+ \times \Omega; X)$;*
- (2) *The family $\mathcal{S}_{\mathcal{K}}^H$ is R -bounded from $L^p(\mathbb{R}_+; \gamma(H, X))$ to $L^p(\mathbb{R}_+ \times \Omega; X)$;*
- (3) *The family $\mathcal{N}_{\mathcal{K}}$ is R -bounded from $L^p(\mathbb{R}_+; X)$ to $L^p(\mathbb{R}_+; \gamma(L^2(\mathbb{R}_+), X))$.*

If X has martingale type 2, the assertions (1)–(3) are equivalent to

- (4) *The family $\mathcal{S}_{\mathcal{K}}$ is R -bounded from $L_{\mathcal{F}}^p(\mathbb{R}_+ \times \Omega; X)$ to $L^p(\mathbb{R}_+ \times \Omega; X)$;*
- (5) *The family $\mathcal{S}_{\mathcal{K}}^H$ is R -bounded from $L_{\mathcal{F}}^p(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ to $L^p(\mathbb{R}_+ \times \Omega; X)$;*

If, moreover, X is a Banach lattice (in which case the type 2 assumption and the martingale type 2 assumption are equivalent), the assertions (1)–(5) are equivalent to

- (6) *The family $\mathcal{T}_{\mathcal{K}^2} := \{T_{k^2} : k \in \mathcal{K}\}$ is ℓ^1 -bounded on $L^{p/2}(\mathbb{R}_+; X^2)$.*

In all equivalences, the R -bounds are comparable with constants depending only on p and X .

Proof. The implications (2) \Rightarrow (1), (4) \Rightarrow (1), (5) \Rightarrow (2), (5) \Rightarrow (4) are trivial, and for Banach lattices X the equivalence (3) \Leftrightarrow (6) is the content of Theorem 5.3.

(2) \Rightarrow (5): Assuming that (2) holds, for any choice of $S_{k_1}, \dots, S_{k_N} \in \mathcal{S}_{\mathcal{K}}^H$ and $G_1, \dots, G_N \in L_{\mathcal{F}}^p(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ we have, by Fubini's theorem and (1),

$$\begin{aligned} \mathbb{E}_r \left\| \sum_{n=1}^N r_n S_{k_n} G_n \right\|_{L^p(\mathbb{R}_+ \times \Omega; X)}^2 &= \mathbb{E}_r \left\| \sum_{n=1}^N r_n S_{k_n} G_n \right\|_{L^p(\mathbb{R}_+; X)}^2 \\ &\leq \rho^2 \mathbb{E}_r \left\| \sum_{n=1}^N r_n G_n \right\|_{L^p(\mathbb{R}_+; \gamma(H, X))}^2 \\ &= \rho^2 \mathbb{E}_r \left\| \sum_{n=1}^N r_n G_n \right\|_{L^p(\mathbb{R}_+ \times \Omega; \gamma(H, X))}^2, \end{aligned}$$

with ρ the R -boundedness constant as meant in (2).

(1) \Leftrightarrow (3): Fix $k_1, \dots, k_N \in \mathcal{K}$ and let G_1, \dots, G_N be elements of $L^p(\mathbb{R}_+; X)$. By Proposition 6.2, for all $t \in \mathbb{R}_+$ we have

$$\mathbb{E}_r \left\| \sum_{n=1}^N r_n (S_{k_n}^H G_n)(t) \right\|_{L^p(\Omega; X)}^p = \mathbb{E}_r \left\| \int_{\mathbb{R}_+} \sum_{n=1}^N r_n k_n(t-s) G_n(s) dW(s) \right\|_{L^p(\Omega; X)}^p$$

$$\begin{aligned}
& \widetilde{\approx}_{p,X} \mathbb{E}_r \left\| s \mapsto \sum_{n=1}^N r_n k_n(t-s) G_n(s) \right\|_{\gamma(L^2(\mathbb{R}_+), X)}^p \\
&= \mathbb{E}_r \left\| \sum_{n=1}^N r_n (N_{k_n} G_n)(t) \right\|_{\gamma(L^2(\mathbb{R}_+), X)}^p.
\end{aligned}$$

An integration over t gives

$$\mathbb{E}_r \left\| \sum_{n=1}^N r_n S_{k_n}^H G_n \right\|_{L^p(\mathbb{R}_+ \times \Omega; X)}^p \widetilde{\approx}_{p,X} \mathbb{E}_r \left\| \sum_{n=1}^N r_n N_{k_n} G_n \right\|_{L^p(\mathbb{R}_+; \gamma(L^2(\mathbb{R}_+), X))}^p.$$

(2) \Leftrightarrow (3): The same argument as in the proof of (1) \Leftrightarrow (3) can be shown that (2) is equivalent with (3)', where

(3)' $\mathcal{N}_{\mathcal{K}}$ is R -bounded from $L^p(\mathbb{R}_+; \gamma(H, X))$ to $L^p(\mathbb{R}_+; \gamma(L^2(\mathbb{R}_+; H), X))$.

The equivalence of (3) and (3)' has been proved in Proposition 5.2. \square

By Theorem 4.7(1), the family $\mathcal{T}_{\mathcal{K}^2}$ is ℓ^1 -bounded if $p/2 > 1$ and the dual of X^2 is an HL lattice (the monotonically completeness assumption is automatically satisfied for dual Banach lattices by [24, Proposition 2.4.19]); recall that $\widetilde{\mathcal{K}} = \{k \in L^1(\mathbb{R}) : |k * f| \leq \widetilde{M}|f| \text{ for all simple } f\}$. Thus we have proved our main result for the stochastic convolution operators S_k :

Theorem 7.2. *Let X be a Banach lattice with type 2 and suppose that the dual of its 2-concavification X^2 is an HL lattice. For all Hilbert spaces H and all $p \in (2, \infty)$, the family of stochastic convolution operators*

$$\{S_k^H : k^2 \in \widetilde{\mathcal{K}}\}$$

is R -bounded from $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ to $L^p(\mathbb{R}_+ \times \Omega; X)$.

Recall that a sufficient condition for X^2 to be an HL space is that X^2 is a UMD Banach function space (see Theorem 4.7(2)).

Note that if $k \in W_{\text{loc}}^{1,1}(\mathbb{R}_+)$ satisfies $\lim_{t \rightarrow \infty} k(t) = 0$ and $\int_0^\infty \sqrt{t} |k'(t)| dt < \infty$, then

$$\begin{aligned}
\int_0^\infty t |(k^2)'(t)| dt &= \int_0^\infty 2t |k'(t)k(t)| dt \\
&= \int_0^\infty 2t |k'(t)| \int_t^\infty k'(s) ds dt \\
&\leq \int_0^\infty 2\sqrt{t} |k'(t)| \int_t^\infty \sqrt{s} |k'(s)| ds dt \\
&\leq 2 \left(\int_0^\infty \sqrt{t} |k'(t)| dt \right)^2
\end{aligned}$$

and therefore $k^2 \in \widetilde{\mathcal{K}}$ by Propositions 4.5 and 4.6. This motivates the following definition:

Let \mathcal{S} be the class of all $k \in W_{\text{loc}}^{1,1}(\mathbb{R}_+)$ such that

$$\lim_{t \rightarrow \infty} k(t) = 0 \quad \text{and} \quad \int_{\mathbb{R}_+} \sqrt{t} |k'(t)| dt \leq 1.$$

The R -boundedness of stochastic convolution with kernels $k \in \mathcal{S}$ was considered in [28, Section 3] in the case $X = L^q$ with $q \in [2, \infty)$.

Note that if $k \in \mathcal{S}$, the above estimate combined with Propositions 4.5 and 4.6 shows that $k^2 \in \widetilde{\mathcal{K}}$. In particular, $k^2 \in L^1(\mathbb{R}_+)$ and therefore $k \in L^2(\mathbb{R}_+)$.

Corollary 7.3. *Let X be a Banach lattice with type 2 and suppose that the dual of its 2-concavification X^2 is an HL lattice. For all Hilbert spaces H and all $p \in (2, \infty)$, the family of stochastic convolution operators*

$$\{S_k^H : k \in \mathcal{S}\}$$

is R -bounded from $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ to $L^p(\mathbb{R}_+ \times \Omega; X)$.

Examples of Banach lattices X satisfying the conditions of the corollaries are the spaces $L^q(S)$ with $q \in [2, \infty)$ (we then have $X^2 = L^{q/2}(S)$).

8. A COUNTEREXAMPLE

It has been an open problem for some time now whether the family

$$\{S_k^H : k^2 \in \widetilde{\mathcal{K}}\}$$

considered in Theorem 7.2 is R -bounded from $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, X))$ to $L^p(\mathbb{R}_+ \times \Omega; X)$ for all $2 < p < \infty$ whenever X is a UMD Banach space with type 2. For UMD Banach lattices X with type 2, by Theorem 7.1 this question is equivalent to asking whether the family

$$\{T_{k^2} : k^2 \in \widetilde{\mathcal{K}}\}$$

is ℓ^1 -bounded on $L^{p/2}(\mathbb{R}_+; X^2)$ for any UMD Banach lattice X of type 2. Here we will prove that this is not the case by showing that the space

$$X = \ell^2(\ell^4)$$

provides a counterexample; for this space we have $X^2 = \ell^1(\ell^2)$ and thus $(X^2)^* = \ell^\infty(\ell^2)$.

Recalling that ℓ^∞ has the HL property, the following result comes somewhat as a surprise:

Proposition 8.1. *The space $\ell^\infty(\ell^2)$ fails the HL property.*

Proof. The proof is a refinement of the argument in [12, Remark 2.9]. Fix an integer $N \geq 1$. Let $f \in L^2(\mathbb{R}; \ell^\infty(\ell^2))$ be defined the coordinate functions

$$(f(t))_j = \mathbf{1}_{(0,1]}(t) \mathbf{1}_{(2^{-j}, 2^{-j+1}]}(t - k2^{-N});$$

the indices k and j stand for the coordinates in ℓ^∞ and ℓ^2 , respectively. Then $\|(f(t))_k\|_{\ell^2} = 1$ for all $t \in (0, 1]$, so $\|f\|_{L^2(\mathbb{R}; \ell^\infty(\ell^2))} = 1$. On the other hand for $1 \leq j \leq N$ and $\tau \in (k2^{-N}, (k+1)2^{-N}]$ with $1 \leq k \leq 2^N - 1$ we have

$$\begin{aligned} \widetilde{M}(f_k)_j(\tau) &= \sup_{r>0} \frac{1}{2r} \left| \int_{\tau-k2^{-N}-r}^{\tau-k2^{-N}+r} \mathbf{1}_{(2^{-j}, 2^{-j+1}]}(t) dt \right| \\ &\geq \frac{1}{2^{-j+2}} \left| \int_{\tau-k2^{-N}-2^{-j+1}}^{\tau-k2^{-N}+2^{-j+1}} \mathbf{1}_{(2^{-j}, 2^{-j+1}]}(s) ds \right| \geq 2^{j-2} \cdot 2^{-j} = \frac{1}{4}, \end{aligned}$$

so

$$\|\widetilde{M}f(t)\|_{\ell^\infty(\ell^2)}^2 \geq \sum_{j=1}^N (\widetilde{M}(f_k)_j(t))^2 \geq \frac{N}{16}, \quad t \in (2^{-N}, 1).$$

Hence $\|\widetilde{M}\|_{L^2(\mathbb{R}; \ell^\infty(\ell^2))} \geq \sqrt{N}/4(1 - 2^{-N})^{1/2}$, which tends to ∞ as $N \rightarrow \infty$. \square

Theorem 8.2. *For any $1 < p < \infty$, the family $\widetilde{\mathcal{T}} = \{T_k : k \in \widetilde{\mathcal{K}}\}$ fails to be ℓ^1 -bounded on $L^p(\mathbb{R}_+; \ell^1(\ell^2))$. As a consequence, for any $2 < p < \infty$ the family*

$$\{S_k^H : k^2 \in \widetilde{\mathcal{K}}\}$$

fails to be R -bounded from $L^p_{\mathcal{F}}(\mathbb{R}_+ \times \Omega; \gamma(H, \ell^2(\ell^4)))$ to $L^p(\mathbb{R}_+ \times \Omega; \ell^2(\ell^4))$.

Proof. By a duality argument, it suffices to show that $\widetilde{\mathcal{T}}$ fails to be ℓ^∞ -bounded on $L^p(\mathbb{R}_+; \ell^\infty(\ell^2))$. As $\ell^\infty(\ell^2)$ fails HL, the latter follows from Proposition 4.10. \square

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